Calculating Numerical Roots

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Introduction – What is a root?

Fig. 1 - HP35s √ key.
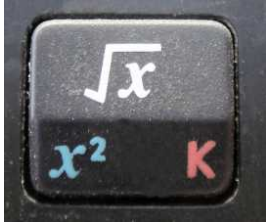
Do you recognize this number[1]?   1.41421 35623 73095 04880 16887 24209 69807 85696 71875 37694 80731 76679 73799 07324 78462 10703 88503 87534 32764 15727 35013 84623 09122 97024 92483 60558 50737 21264 41214 97099 93583 14132 22665 92750 55927 55799 95050 11527 82060 57147 01095 59971 60597 02745 34596 86201 47285 17418 64088 91986 09552 32923 04843 08714 32145 08397 62603 62799 52514 07989 68725 33965 46331 80882 96406 20615 25835 23950 54745 75028 77599 61729 83557 52203 37531 85701 13543 74603 40849 …

If you have had any math classes you have probably run into this number and you should recognize it as the square root of two, $\sqrt{2}$. The square root of a number, n, is that value when multiplied by itself equals n. 2 x 2 = 4 and $\sqrt{4}$ = 2. Calculating the square root of the number is the inverse operation of squaring that number $\sqrt{n^2}$ = n. The "√" symbol is called the "radical" symbol or "check mark."

MS Word has the radical symbol, √, but we often use it with a line across the top. This is called the "vinculum" circa 12th century. The expression "$\sqrt{n}$[2]" is read as "root n", "radical n", or "the square root of n". The vinculum extends over the number to make it inclusive e.g. $\sqrt{98,696,044,010.9}$ = 314,159.265359. Most HP calculators have the square root key as a primary key as shown in Fig. 1 because it is used so much.

Roots and Powers

Numbers may be raised to any power (y, multiplied by itself x times) and the inverse operation, taking the root, may be expressed as shown below.

$$\sqrt[4]{81} = 3 \qquad 81 = 3 \times 3 \times 3 \times 3$$

The problem is taking the fourth root of 81. This problem may also be expressed as:

Fig. 2 – HP 35s Power Key.

$$81^{1/4} = 3 \qquad \text{Raising a number to a power uses the key shown in Fig. 2.}$$

The key sequence is 81, ENTER, 4, 1/x, y^x. The "extra" key is the reciprocal key, 1/x.

Without your HP calculator you would have a very difficult time making this calculation. This problem is easy because it uses integers. How about the problem of:

$$\sqrt[\pi]{e} = {}^{3.14159265359}\sqrt{2.71828182846} = 1.37480222744.$$

As a quick check, with the answer displayed, press ENTER twice and x twice to get 2.59848779509 which is the answer cubed. This is very close to the number under the radical. The root is a little over 3.

Using the y^x key for square roots is good practice for all power and root problems. Solving the example problem given above, $\sqrt{98,696,044,010.9}$ = 314,159.265359, you would key 98696044010.9, ENTER, 2,

1/x, y^x. This is an extra key press for square roots, but it will give you a generalized procedure which provides confidence when the more complex problems like $^\pi\sqrt{e}$ present themselves.

The HP 35s is our "standard" calculator for the Fundamentals of Applied Math Series unless specified otherwise. The suggested general process recommend above is actually for most other models. The key shown in Fig. 1 also has a left shifted function $^x\sqrt{y}$ as shown in Fig. 3. This should be used for calculating numerical roots. The number of keystrokes is the same because the root key is a shifted key.
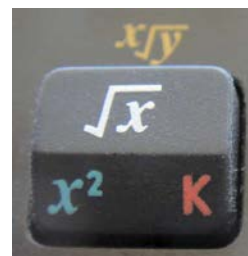
Fig. 3 – HP 35s General Root Key.

Negative Roots

If the square root of a number, m, is that number, n, that when multiplied by itself equals the number, what happens when the number, m, is negative? Two negative numbers, when multiplied, results in a positive number. Is the square (even) root of a negative number meaningless? The cube (odd) root of -27 has a solution, -3, because -3 x -3 x -3 = -27.

Taking a square root of a negative number on the HP 35s results in a "SQRT(NEG)" message. Taking the even root, of a negative number on most HP calculators using y^x will usually result in an ERROR message (INVALID on the HP-35).

For a very long time the idea of taking the square root of a negative number couldn't be represented in terms of "normal" mathematical operations – until the concept of the imaginary unit, often called a unit imaginary number was applied. The imaginary unit is usually designated as i[3]. No real number has a negative square so the (confusing) term imaginary is used.

Remembering that a negative multiplied by a negative results in a positive, the square root of any real number, excluding zero, has two values ±. The term often applied to these roots is the principle, +, root.

Let's use the definitional property of i.

$$i^2 = -1$$

In practical terms $i = \sqrt{-1}$.

The square root of -1 has two values, $+i$ and $-i$. While these two values are the negatives of each other, they are algebraically the same.

The square root of -2 is expressed as 1.41421356237i which is an imaginary number and a particular case of complex numbers. The subject of complex numbers will be covered in a future installment of Fundamentals of Applied Math Series.

Imaginary numbers may be operated on in a similar manner as real numbers. Squaring an imaginary number is a negative number (zero excluded). The HP 35s has operations to perform complex arithmetic (+, -, x, ÷), complex trigonometry (sin, cos, tan) and the mathematical functions –z, 1/z, Z^{z2}, ln z, and e^z - where z and z2 are complex numbers.

The square root of -2 is also expressed on more advanced calculators as a complex number which is an ordered pair of numbers shown as (0,1.41421356237). The i is assumed to be attached to the second number. Most complex number functions are found on algebraic and graphing calculators.

Table 1 shows the current HP calculator lineup. All of the financial, scientific, and graphing calculators have a $\sqrt{}$, and a $y^{x(4)}$ key. The OfficeCalc[(5)] 100, OfficeCalc 200, and OfficeCalc 300 has a $\sqrt{}$ key, but no y^x key. Some of the Scientific and Graphing machines also have a symbolic capability which is indicated by an *.

Table 1 – Current HP Calculator Products (24)

#	Financial Calculators	Scientific & Graphing	Home & Office
1	HP 10bII	HP 10s	HP Calcpad 100
2	HP 10bII+	HP-15C Limited Edition	HP CalcPad 200
3	HP 20b	HP 33s	OfficeCalc 100
4	HP 30b	HP 35s	OfficeCalc 200
5	HP-12C	HP 39gs*	OfficeCalc 300
6	HP-12C 30th Anniversary	HP 39gII* (new)	PrintCalc 100
7	HP-17bII+	HP 40gs*	QuickCalc
8		HP 48gII*	
9		HP 50g*	
10		SmartCalc HP 300s	

* *indicates a symbolic math capability.*

The symbolic features (complex operations) of these machines allow them to deal with negative roots.

Imaginary Numbers

i is a number and not a variable. i is not found on the real number line because *i* is not a real number.

When dealing with powers of *i* you must remember four cases that repeat themselves in a cycle.

Table 2 – Cyclic 4 Values of Powers of i

$i^{-4} = 1$	$i^0 = 1$	$i^4 = 1$
$i^{-3} = i$	$i^1 = i$	$i^5 = i$
$i^{-2} = -1$	$i^2 = -1$	$i^6 = -1$
$i^{-1} = -i$	$i^3 = -i$	$i^7 = -i$

Imaginary numbers require care in their use. Use the same rules you normally do. Here are some examples.

$$i^2 = \sqrt{-1} \cdot \sqrt{-1} = -1$$

$$i^3 = i^2 \cdot i = -1 \cdot i = -i$$

$$\sqrt{-7} = \sqrt{-1 \cdot 7} = \sqrt{-1} \cdot \sqrt{7} = i\sqrt{7}$$

$$\sqrt{-a} = \sqrt{-1 \cdot a} = \sqrt{-1} \cdot \sqrt{a} = i\sqrt{a}$$

Here is an example of multiplying square roots that is incorrect because $\sqrt{x}\sqrt{y} = \sqrt{xy}$ is only true if at least one of the two numbers (x & y) is positive. $i^2 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$

Observations and Conclusions

Calculating numerical roots is a challenging problem if you did not have a modern electronic calculator. We learn many roots in school, especially the squares of the numbers 1 to 12 so that the square root of 144 is (or should be) from memory to be 12. We normally think of square roots, but any-number root, is mathematically possible. A problem like $^{\pi}\sqrt{e} = {}^{3.14159265359}\sqrt{2.71828182846} = 1.37480222744$ would not be thinkable without a calculator. Some root problems such as an even root of a negative number is not possible, e.g. $^{6}\sqrt{-13}$ unless expressed as an imaginary number, i. In practical terms $i = \sqrt{-1}$. Most calculators will give an ERROR message but more advanced (algebraic) calculators will calculate (even roots of negative numbers) using complex numbers. Complex numbers will be discussed in a future Fundamentals of Applied Math installment.

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Calculating Roots - Notes

(1) Three hundred ninety five digits are shown here for the irrational square root of two See the link below for one million digits and a referenced link for 10 million digits.   http://apod.nasa.gov/htmltest/gifcity/sqrt2.1mil If the digits are downloaded it requires 261 pages, is 1,000,158 digits, and 12,502 lines. The file size is 993 KB, and 688 KB as a pdf file. Most five digit sequences will be found if the $\sqrt{2}$ digits are searched e.g. the first five digits of π (7x), e (5x), φ (13x). The square root of 2 has some unique and interesting properties. See how it may be calculated geometry and used to approximate π as well as other properties at: http://en.wikipedia.org/wiki/Square_root_of_2

(2) Adding mathematics expressions to text is always difficult. While there are many options for doing this one common solution is to create an image and insert it into the document. One technique is to type it slightly larger as closely as possible in MS Word and screen capturing the expression into an image program. The image is then touched up, and converted to a tightly cropped B&W image. The image may then be sized as desired when inserted into MS Word. This process produces a more bold looking set of characters.

(3) Electrical engineers will use j as the imaginary unit because i is used for electrical current.

(4) Finical calculators use the math operations behind the y^x key making it easy to include this very powerful function on the keyboard.

(5) The officeCalc series of calculators are 14 digit calculators. These machines will provide the greatest number of digits for calculating square roots. Programs running on the programmable models will provide square roots of numbers to hundreds of digits – usually limited by memory.

About the Author

Richard J. Nelson has written hundreds of articles on the subject of HP's calculators. His first article was in the first issue of *HP 65 Notes* in June 1974. He became an RPN enthusiast with his first HP Calculator, the HP-35A he received in the mail from HP on July 31, 1972. He remembered the HP-35A in a recent article that included previously unpublished information on this calculator. See http://hhuc.us/2007/Remembering%20The%20HP35A.pdf
He has also had an article published on HP's website; HP Calculator Firsts.
See http://h20331.www2.hp.com/Hpsub/cache/392617-0-0-225-121.html.