Continued Fractions: A Step-by-Step Tutorial, with User RPL Implementations

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Chapter I

How to convert decimal fractions & regular fractions into continued fractions.

Part A: Step-by-step example.

Example: Convert the terminating decimal 3.1416 into its exactly equivalent continued fraction.

Step 1: Rewrite 3.1416 as an equivalent simple fraction: $\frac{31416}{10000}$. Reduce it: $\frac{3927}{1250}$. Step 2: Rewrite that as a mixed number: $3 + \frac{177}{1250}$. Step 3: Remove the integer part: $\frac{177}{1250}$. Step 4: Reciprocate: $\frac{1250}{777}$. Repeat steps 2 through 4 until no fraction part remains: $\Rightarrow \frac{1250}{777} = 7 + \frac{11}{177}$.

$$\rightarrow \frac{177}{11} = \mathbf{16} + \frac{1}{11}.$$

 \rightarrow 11, stop.

Step 5: Write all the integer parts (in red above) in a list: {3, 7, 16, 11}. Done.

Answer: 3.1416 is exactly equivalent to the continued fraction {3, 7, 16, 11}.

Answer: 3.1410 10 cm. Note: The notation "{3, 7, 16, 11}" is mathematical shorthand for $3 + \frac{1}{7 + \frac{1}{16 + \frac{1}{11}}}$.

Part B: Generalized method.

To convert a decimal fraction or a simple fraction into a continued fraction:

Step 1: If you have a decimal fraction, rewrite it as a simple fraction. Reduce the simple fraction.

Step 2: Rewrite it as a mixed number.

Step 3: Remove the integer part.

Step 4: Reciprocate.

Repeat steps 2-4 until no fraction part remains.

Step 5: Write all the integer parts in a list. Done.

Part C: RPL implementation.

Step 1: Convert a decimal fraction into an equivalent reduced Simple Fraction:

« IF DUP FP THEN 11. OVER XPON - ALOG SWAP OVER * R→I SWAP R→I SIMP2 END » DUP '→SF' STO BYTES → 65.5 #94E9h

This creates a program called $\rightarrow SF'$ ("to <u>Simple Fraction</u>"). Input: Any decimal fraction (a floating point number w/ non-zero fractional part). Output: Equivalent Simple Fraction.

Here's a method for doing steps 2 through 5 manually:

Step 2: Rewrite the simple fraction as a mixed number: **PROPFRAC**. Write down the integer part.

Step 3: Remove the integer part: $\text{``DUP IP } R \neq I - EVAL \text{```} (or just subtract the integer that you see in the display, and simplify by pressing EVAL).$

Step 4: Reciprocate: INV (the $\sqrt[1]{x}$ key).

Repeat steps 2 through 4 until no fraction part remains.

Step 5: Write the accumulated integer parts in a list. Done.

Here's a User RPL program for doing steps 2 through 5 automatically. It converts any simple fraction into its equivalent <u>Continued Fraction in one step</u>:

« FXND DEPTH DUP 1 + ROLLD DO SWAP OVER IDIV2 ROT SWAP DUP UNTIL 0 == END DROP2 DEPTH ROLL DEPTH SWAP - 1 + →LIST » DUP '→CF' STO BYTES → 81 #C386h

This creates a program called $' \Rightarrow CF'$ ("to <u>C</u>ontinued <u>F</u>raction"). Input: Any simple fraction (as an algebraic object). Output: Equivalent <u>C</u>ontinued <u>F</u>raction (as a list).

Example: '13/46' \rightarrow CF \rightarrow (0 3 1 1 6)

This means that $\frac{13}{46} = 0 + \frac{1}{3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{6}}}}$.

Use both programs in sequence to convert a decimal fraction into a continued fraction.

Example: 3.1416 \Rightarrow SF \Rightarrow CF \rightarrow (3 7 16 11)

This means that
$$3.1416 = 3 + \frac{1}{7 + \frac{1}{16 + \frac{1}{11}}}$$

Chapter II

How to convert continued fractions into simple fractions & decimal fractions.

Part A: Step-by-step example.

Example: Convert the continued fraction {3 7 16 11} into its equivalent simple fraction and decimal fraction.

Step 1: Put list on the stack and press **EWAL**. Four stack levels are now occupied.

Step 2: Press (yx) (1) represented by the full notation $3 + \frac{1}{7 + \frac{1}{16 + \frac{1}{11}}}$ Step 2: Press (\sqrt{x}) (+) repeatedly until all the numbers are used up. This converts the shorthand

Step 3: Press **EVAL** and see the answer: $\frac{3927}{1250}$.

Optional Step 4: To convert that into a decimal fraction, press \rightarrow UM and see the answer: 3.1416.

Part B: Generalized method.

To convert a continued fraction into a simple fraction or a decimal fraction:

Step 1: Put the continued fraction on the stack in list form and press **EWAL**. Step 2: Press (1/x) (+) repeatedly until all the numbers are used up. Step 3: Press **EWAL**. See the equivalent simple fraction. Optional Step 4: Press **HUM**. See the equivalent decimal fraction.

Part C: RPL implementation.

```
≪ LIST→ 1 0 1 4 ROLL
  START OVER 4 ROLL * + SWAP
  NEXT / »
DUP 'CF→' STO
BYTES → 47.5 #1900h
```

This creates a program called ${}^{\mathsf{CF}}$ ("Continued Fraction out"). Input: Any Continued Fraction (as a list). Output: Equivalent simple fraction.

To convert the output to a decimal fraction, just press \rightarrow HUH, of course.

Example: Convert the continued fraction {3 7 16 11} into its equivalent simple fraction and decimal fraction.

$(3 7 16 11) CF \rightarrow \frac{3927}{1250}$ $\rightarrow NUM \rightarrow 3.1416.$

Chapter III Fun application: Opinion Polls

Newspapers usually report the results of opinion polls as percentages. For example, they might report that 42.86% of the people in a recent opinion poll responded "Yes." This looks impressive, subconsciously implying that many people were polled. Of the very few readers for whom that thought actually bubbles into their awareness, most would rationalize that feeling thus: "42.86% means that 4286 out of every 10000 people responded Yes," implying that ten thousand people were polled (if not more). Of those few readers, fewer still would take the further step of reducing the fraction $\frac{4286}{10000}$ to the equivalent fraction $\frac{2143}{5000}$, thus recognizing that as few as 5000 people may have been polled, with 2143 of them responding "Yes."

However, none of these conclusions are correct, because they ignore the fact that statistics are *rounded off.* What the newspaper is really saying, therefore, is that the percentage of Yes responses may not have been exactly 42.86% at all, but was something which merely *rounds* to 42.86%. Therefore, the *actual* percentage could have been anything between 42.855% (inclusive) and 42.865% (exclusive).

So the question becomes: What is the *smallest* number of respondents which can make such a ratio possible? In other words, what is the "simplest fraction" (the ratio of the smallest possible numbers) in the interval [0.42855, 0.42865)?

The only way to solve that problem is with continued fractions.

First, convert both ends of the interval into a continued fraction. Let's use the programs listed above.

.42855 →SF →CF → (0 2 2 1 951 1 2) .42865 →SF →CF → (0 2 3 259 2 5)

The lists are the same until the third terms. Truncate the list there, *using the smaller number* (2 *here*) *plus 1:*

```
(0 2 <mark>3</mark>) CF→ → '3/7'
→NUM → .428571428571
```

As you can see, 3/7 rounded to four decimal places is .4286, so the statistic "42.86%" could have been achieved with a polling sample of only *SEVEN* people, of whom 3 responded "Yes".

Example #2:

We asked all of the students at Wassamada University whether they preferred vanilla or chocolate. Those who preferred neither were forced to choose one anyway. 68.421% of the students responded that they preferred chocolate. How many students attend Wassamada U? Ten thousand, as the statistic seems to imply? Not necessarily. What is the "simplest fraction" in the interval [.684205, .684215)?

```
.684205 →SF →CF → (0 1 2 6 501 10 2)

.684215 →SF →CF → (0 1 2 5 1 618 2 1 5)

Truncate the list where they begin to differ, using the smaller number (5 here) plus 1:

(0 1 2 6) CF → \rightarrow '13/19'

\frac{13}{19} is therefore the simplest fraction that rounds to 68.421%. Therefore it's statistically possible

that Wassamada U has only 19 students, of whom 13 prefer chocolate.
```

The rule "Truncate the lists where they begin to differ and use the smaller number plus one" might sound peculiar. Here's a good example of that rule in action: What is the "simplest fraction" between $\frac{21}{31}$ and $\frac{41}{61}$, non-inclusive? '21/31' \rightarrow CF \rightarrow (0 1 2 10) '41/61' \rightarrow CF \rightarrow (0 1 2 20) Truncate the list where they begin to differ, and use the smaller number (10 here) *plus 1:* (0 1 2 11) CF \rightarrow '23/34' Therefore $\frac{23}{34}$ is the "simplest fraction" between $\frac{21}{31}$ and $\frac{41}{61}$, that is to say, every other fraction in that interval has a numerator larger than 23 and a denominator larger than 34.

Homework: Find the simplest fraction between π and $\sqrt{10}$. Hint: you already know it.

Chapter IV	
Glossary	

Fraction: The ratio of two numbers; the relation between two similar magnitudes with respect to the number of times the first contains the second.

Common Fraction: a fraction represented as a numerator above, and a denominator below, a horizontal line, e.g. $\frac{\pi}{\phi}$, or a diagonal line, e.g. $\frac{\pi}{\phi}$ or π/ϕ . The line is known as a "virgule", "solidus", "slash", or "stroke", but is usually read as "over" (e.g. $\frac{\pi}{\phi}$ is read "pi over phi") or "divided by" ("pi divided by phi"). Children are taught to use a special "division sign" instead of a slash (e.g. $6 \div 2 = 3$), but this sign is rarely used in higher math, even though it appears on the division key of even the most advanced handheld calculators. Children are also taught to 63

represent $126 \div 2 = 63$ as $2\overline{)126}$, and read it as "2 goes into 126, 63 times."

Decimal Fraction: a fraction whose denominator is some power of 10, and is represented by a dot ("decimal point" or "point") (or a comma in Europe and some other places) written with the numerator, e.g. 0.75 which is equivalent to the simple fraction $\frac{75}{100}$. In RPL calculators, decimal fractions are called "reals". In common parlance they are called "floating point numbers".

Simple Fraction: the ratio of an integer to a natural number, e.g. $\frac{22}{7}$. According to this strict definition, the numerator may be zero or negative, but not the denominator.

Compound Fraction: a fraction whose numerator and/or denominator contain one or more fractions.

Integer: A number with no fractional part; a member of the set {... -2, -1, 0, 1, 2, ...}.

Whole Number: A non-negative integer; a member of the set $\{0, 1, 2, ...\}$.

Natural Number: A non-zero whole number; a member of the set $\{1, 2, 3, ...\}$, which is also known as the set of "Counting Numbers".

Proper Fraction: a fraction whose numerator is less than the denominator, e.g. $\frac{7}{11}$.

Improper Fraction: a fraction whose numerator is greater than the denominator, e.g. $\frac{11}{7}$.

Mixed Number: the sum of an integer and a proper fraction, e.g. $3 + \frac{1}{7}$.

Continued Fraction: a compound fraction whose denominator contains a fraction whose 2

denominator contains a fraction, and so on, e.g. $1 + \frac{2}{3 + \frac{4}{5}}$ which is equivalent to the simple

fraction $\frac{29}{19}$. Continued fractions can contain a finite or infinite number of fractions.

Simple Continued Fraction: a continued fraction all of whose numerators are 1, e.g.

 $5 + \frac{1}{3 + \frac{1}{7}}$. Mathematical shorthand notation for simple continued fractions is a list of just the

integer parts, ignoring the numerators which are always 1; the above example would be written as $\{5, 3, 7\}$.

Rational Number: A number which can be represented as a simple fraction, e.g. the terminating decimal 0.5 which can be represented as $\frac{1}{2}$, or the repeating decimal $0.\overline{3}$ which can be represented as $\frac{1}{3}$. All rational numbers can be represented as either a terminating decimal or a repeating decimal.

Irrational Number: A number which cannot be represented as a simple fraction, e.g. π , which is not exactly equal to any simple fraction. When represented as *decimal fractions*, irrational numbers neither terminate nor repeat. However, the *continued fraction* representation of some irrational numbers repeat (e.g. the square roots of non-square whole numbers); some contain some other non-repeating but obvious pattern (e.g. Euler's constant *e*); and some seem utterly patternless (e.g. π).

Terminating Decimal: A decimal fraction which has a finite number of non-zero digits, e.g. 0.125 (equal to $\frac{1}{8}$). Also known as a "finite decimal."

Non-terminating Decimal: A decimal fraction which contains an infinite number of non-zero digits, e.g. π or $\frac{1}{7}$, neither of which terminate when written as a decimal fraction. Also known as an "infinite decimal."

Repeating Decimal: a decimal numeral that, after a certain point, consists of a group of one or more digits repeated ad infinitum, e.g. $1.2\overline{34}$ (with the overscore meaning that those digits repeat forever; this example is equal to $\frac{611}{495}$, and can also be represented as 1.2343434..., with the ellipsis meaning "and so on"). Also known as a "circulating decimal", "periodic decimal", and "recurring decimal."

Simplest Fraction: The unique simple fraction with the smallest numerator and denominator which lies within a given interval. For example, the simplest fraction within the interval $\pi \pm 0.0005$ is $\frac{267}{85}$, because all the other simple fractions in that interval have larger numerators and denominators. The simplest fraction in the interval [.4, .6] is obviously ¹/₂. The simplest fraction in the interval [.12345, .12346] is $\frac{10}{81}$, which is less obvious, but is easily calculated using continued fractions, as shown in Chapter III above.

PDQ Algorithm: A method for calculating the simplest fraction in any given interval, performing all calculations on the integer domain, thus avoiding all round-off errors. Discovered by Joseph Horn (impatient Number Theory student), and optimized by Rodger Rosenbaum (patient Number Theory master) and Tony Hutchins (inveterate RPL programmer). It solves the Wassamada U example above in one step, like this:

.68421 .000005 PDQ → '13/19'