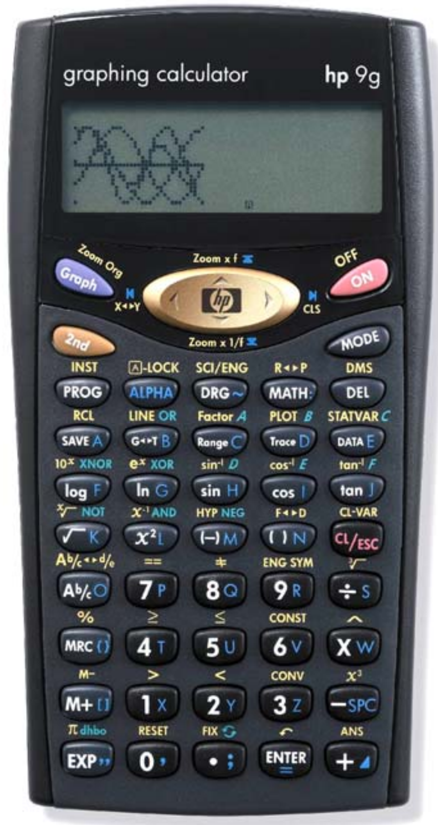
hp calculators

HP 9g Solving Problems Involving Complex Numbers



Basic Concepts

Practice Solving Problems Involving Complex Numbers

Practice Example: Roots of a Cubic Equation

Basic concepts

There is *no* real number *x* such that $x^2 + 1 = 0$. To solve this kind of equations a new set of numbers must be introduced. A complex number is a number of the form $a + bi$ where *a* and *b* are real numbers and *i* is the square root of –1, i.e. $i^2 = -1$, and is called the imaginary unit. Since *i* is used for representing the intensity of current in electromagnetism, engineers often write the imaginary unit as *j*. The real *a* is called the real part of the complex number, and *b*, also real, is the imaginary part. When both *a* and *b* are integers, the complex number $a + bi$ is called a Gaussian integer (e.g. $-4 + 3i$). Notice that real numbers can be thought as the subset of complex numbers whose imaginary part is zero. Here are the most basic rules:

- $a + bi = c + di$ if and only if a = c and b = d
- $(a + bi) + (c + di) = (a + c) + (b + d)i$
- $(a + bi) \times (c + di) = (ac - bd) + (ad + bc)i$
- $\dfrac{a + bi}{c + di} = \dfrac{ac + bd}{c^2 + d^2} + \dfrac{bc - ad}{c^2 + d^2} i$

The conjugate complex number of $a + bi$ is $a - bi$. Note that the product of a pair of conjugate numbers is a real number ($a^2 + b^2$). The modulus or absolute value of the complex number $a + bi$ is defined as $\sqrt{a^2 + b^2}$.

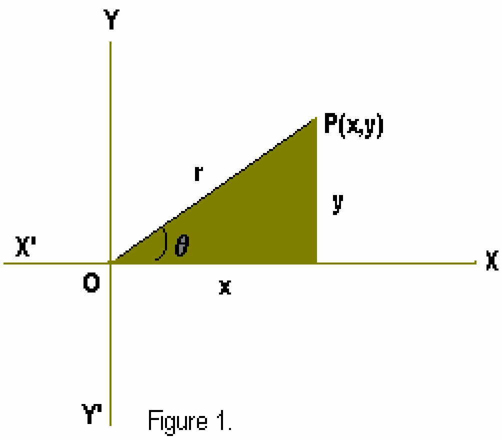


Figure 1.

Figure 1 shows the complex plane, also known as the *Argand diagram*. It is a representation of the complex number $z = x + yi$. XX' is the real axis, and YY' is the imaginary axis. The point P whose cartesian coordinates are (x, y) is called the affix of the complex number z. Note that r (the distance of the affix from the origin O) is equal to the modulus of z. The angle θ is called the argument of z. From the figure: $\tan\theta = \dfrac{y}{x}$

A complex number $x + yi$ can be represented in various ways:

- (x, y) i.e. as cartesian coordinates – the rectangular form.
- $r\cos\theta + ir\sin\theta$. This is the modulus-argument form or polar form, often written as $(r, \angle\theta)$, i.e. the polar coordinates.
- $re^{i\theta}$. It is the exponential form, and is based on Euler's formula $\cos\theta + i\sin\theta = e^{i\theta}$ (where θ is expressed in radians). The function $\cos\theta + i\sin\theta$ is sometimes referred to as cis θ. Note that if $\theta = \pi$ then we obtain the well-known identity $e^{i\pi} = -1$.

Practice solving problems involving complex numbers

The HP 9g has no specific functions for operating with complex numbers. Having said that, this calculator is powerful enough to carry out calculations with complex numbers easily. It is the purpose of this learning module to introduce some methods of doing such calculations on your HP 9g.

Example 1: Find the modulus and the argument of the complex number 5 + 6·i.

Solution: The R↔P menu ([2nd] [R↔P]) contains four functions that are very useful when operating with complex numbers. They are the rectangular to and from polar conversion functions, which are described in greater detail in the HP 9g learning module *Polar/Rectangular Coordinate Conversions*. For our purpose, R ▶ Pr and R ▶ Pθ return the modulo and the argument, respectively, of a complex number expressed in rectangular form. The real and imaginary parts are separated by commas ([ALPHA] [0 ,]), e.g. R ▶ Pr(a,b). In this example, to find the modulus press:

[2nd] [R↔P] [ENTER] [5 U] [ALPHA] [0 ,] [6 V] [ENTER]

and to find the argument press:

[2nd] [R↔P] ▶ [ENTER] [5 U] [ALPHA] [0 ,] [6 V] [ENTER]

Remember that the argument of a complex number is an angle, and therefore its value depends on the angle unit. The R ▶ Pθ function always returns the angle in the current mode.

Answer: Rounding to four decimal digits, r = 7.8102 and θ = 50.1944º

Example 2: The voltage in a circuit is 45 + 5j volts and the impedance is 3 + 4j ohms. Find the total current.

Solution: The current is given by the following formula:

$$I = \frac{E}{Z} = \frac{45+5j}{3+4j}$$

We therefore have to divide two complex numbers. One way of doing this calculation is to use the basic formula given above, which divides two complex numbers expressed in rectangular form. While additions and subtractions of complex numbers are easily done in rectangular form, the product and division are much easier if the numbers are in exponential form because:

$$re^{j\theta} \cdot qe^{j\varphi} = rq \cdot e^{j(\theta+\varphi)}$$

To multiply (divide) two complex numbers we just have to multiply (divide) their moduli and add (subtract) their arguments. But, be warned that *<u>you must work in radians</u>* when using this formula. In our case, the modulus of the current will be the modulus of E divided by the modulus of Z, i.e. R ▶ Pr(45,5) ÷ R ▶ Pr (3,4). To store this modulus in the variable R press:

[2nd] [R↔P] [ENTER] [4 T] [5 U] [ALPHA] [0 ,] [5 U] ▶ [÷ S] [2nd] [R↔P] [ENTER] [3 Z] [ALPHA] [0 ,] [4 T] [ENTER] [SAVE A] [9 R] .

The argument of the current is simply the difference between the arguments: R ▶ Pθ(45,5) – R ▶ Pθr (3,4). Store the resulting argument in T by pressing:

[2nd] [R↔P] ▶ [ENTER] [4 T] [5 U] [ALPHA] [0 ,] [5 U] ▶ [– SPC] [2nd] [R↔P] ▶ [ENTER] [3 Z] [ALPHA] [0 ,] [4 T] [ENTER] [SAVE A] [4 T] .

Now we have to convert the current vector to rectangular form using these formulae: P ▸ Rx(R, T) and P ▸ Ry(R,T), i.e.

[2nd] [R↔P] ▼ [ENTER] [2nd] [ALPHA] [9 R] [0 ,] [4 T] [ALPHA] [ENTER] and
[2nd] [R↔P] ▲ [ENTER] [2nd] [ALPHA] [9 R] [0 ,] [4 T] [ALPHA] [ENTER] respectively.

Answer: $9.0554e^{-j0.8166} = 6.2 - 6.6j$ amperes.

Practice example: roots of a cubic equation

Programs can be written to make all these calculations easier. In this example we will write a program to find the roots of a cubic equation, including the complex ones. While it is a useful application by itself, the purpose of this example is to show the way you can tackle non-trivial problems involving complex numbers on your HP 9g, including a way to display complex results. In brief, the method we'll use to find the roots is as follows. We'll solve the general equation $x^3 + ax^2 + bx + c = 0$. Note that any cubic equation can be reduced to this form by dividing each coefficient by the original coefficient of x^3. Then, we need to calculate:

$$d = \frac{-a^2}{3} + b \qquad e = (\frac{6a^2}{27} - b)\frac{a}{3} + c \qquad f = e^2 + \frac{4d^3}{27} \qquad g = \sqrt[3]{\frac{-e + \sqrt{f}}{2}} \qquad h = \sqrt[3]{\frac{-e - \sqrt{f}}{2}}$$

If **f** > 0 then there's one real root $x_1 = g + h - \frac{a}{3}$ and a complex conjugate pair, whose real part is $-(\frac{g+h}{2} + \frac{a}{3})$ and the imaginary part is $\pm\sqrt{3}\,\frac{g-h}{2}i$. If **f = 0** then there are three real roots with at least two equal:

$x_1 = x_2 = -(\frac{g+h}{2} + \frac{a}{3})$ and $x_3 = g + h - \frac{a}{3}$. If **f** < 0 then there are three real but unequal roots: let

$\phi = \frac{1}{3}\cos^{-1}(\frac{-e}{2}\sqrt{\frac{-d^3}{27}})$, then $x_1 = \frac{1}{3}(\sqrt{\frac{-d}{3}}6\cos(\phi) - a)$, $x_2 = \frac{1}{3}(\sqrt{\frac{-d}{3}}6\cos(\phi + 120^o) - a)$ and

$x_3 = \frac{1}{3}(\sqrt{\frac{-d}{3}}6\cos(\phi + 240^o) - a)$

Here's the program. It's 376 steps long, so make sure there are enough program steps available before typing it in. The table below includes the keystrokes needed to enter the program once you are in Program Edit mode. (Refer to the learning module *Writing a Small Program* for more information on creating and editing programs).

Line	Commands	Keys
Line 1	INPUT A	[2nd] [INST] ▲ ▲ ▲ [0 ,] [ALPHA] [SAVE A] [ENTER]
Line 2	INPUT B	[2nd] [INST] ▲ ▲ ▲ [0 ,] [ALPHA] [G↔T B] [ENTER]
Line 3	INPUT C	[2nd] [INST] ▲ ▲ ▲ [0 ,] [ALPHA] [Range C] [ENTER]
Line 4	D=-A²/3+B	[ALPHA] [Trace D] [ALPHA] [ENTER] [(−) M] [ALPHA] [SAVE A] [x² L] [÷ S] [3 Z] [+ ◢] [ALPHA] [G↔T B] [ENTER]
Line 5	E=(A/3)(6A²/27–B)+C	[ALPHA] [DATA E] [ALPHA] [ENTER] [( ) N] [ALPHA] [SAVE A] [÷ S] [3 Z] ▶ [( ) N] [6 V] [ALPHA] [SAVE A] [x² L] [÷ S] [2 Y] [7 P] [− SPC] [ALPHA] [G↔T B] ▶ [+ ◢] [ALPHA] [Range C] [ENTER]

Line 6	F=E^2+4D^3/27	[ALPHA] [log F] [ALPHA] [ENTER] [ALPHA] [DATA E] [x^2 L] [+] [4 T] [ALPHA] [Trace D] [2nd] [x^3] [÷ S] [2 Y] [7 P] [ENTER]
Line 7	IF(F<0)THEN{	[2nd] [PROG] [ENTER] [ALPHA] [log F] [2nd] [<] [0 ,] ▶ ▶ [ENTER]
Line 8	PRINT “3 REAL”	[2nd] [PROG] ▲ ▲ ▲ [ENTER] [ALPHA] [EXP] [3 Z] [2nd] [ALPHA] [− SPC] [9 R] [DATA E] [SAVE A] [x^2 L] [ALPHA] ▶ [ENTER]
Line 9	SLEEP(1)	[2nd] [PROG] ▲ [0 ,] [1 X] ▶ [ENTER]
Line 10	I=(1/3)cos^{-1}((-E/2)√(-D^3/27))	[ALPHA] [cos I] [ALPHA] [ENTER] [( ) N] [1 X] [÷ S] [3 Z] ▶ [2nd] [$\cos^{-1}$ E] [( ) N] [(−) M] [ALPHA] [DATA E] [÷ S] [2 Y] ▶ [√ K] [(−) M] [ALPHA] [Trace D] [2nd] [x^3] [÷ S] [2 Y] [7 P] ▶ ▶ [ENTER]
Line 11	X=(1/3)(√(-D/3)6cos(I)–A)	[ALPHA] [1 X] [ALPHA] [ENTER] [( ) N] [1 X] [÷ S] [3 Z] ▶ [( ) N] [√ K] [(−) M] [ALPHA] [Trace D] [÷ S] [3 Z] ▶ [6 V] [cos I] [ALPHA] [cos I] ▶ [− SPC] [SAVE A] ▶ [ENTER]
Line 12	Y=(1/3)(√(-D/3)6cos(I+120)–A)	[ALPHA] [1 X] [ALPHA] [ENTER] [( ) N] [1 X] [÷ S] [3 Z] ▶ [( ) N] [√ K] [(−) M] [ALPHA] [Trace D] [÷ S] [3 Z] ▶ [6 V] [cos I] [ALPHA] [cos I] [+] [1 X] [2 Y] [0 ,] ▶ [− SPC] [SAVE A] ▶ [ENTER]
Line 13	Z= (1/3)(√(-D/3)6cos(I+240)–A)	[ALPHA] [1 X] [ALPHA] [ENTER] [( ) N] [1 X] [÷ S] [3 Z] ▶ [( ) N] [√ K] [(−) M] [ALPHA] [Trace D] [÷ S] [3 Z] ▶ [6 V] [cos I] [ALPHA] [cos I] [+] [2 Y] [4 T] [0 ,] ▶ [− SPC] [SAVE A] ▶ [ENTER]
Line 14	PRINT X; ◢	[2nd] [PROG] ▲ ▲ ▲ [ENTER] [ALPHA] [1 X] ▶ [ALPHA] [+] [ENTER]
Line 15	PRINT Y; ◢	[2nd] [PROG] ▲ ▲ ▲ [ENTER] [ALPHA] [2 Y] ▶ [ALPHA] [+] [ENTER]
Line 16	PRINT Z	[2nd] [PROG] ▲ ▲ ▲ [ENTER] [ALPHA] [3 Z] [ENTER]
Line 17	END }	[2nd] [PROG] ▲ [1 X] [ALPHA] ▶❙ [ENTER]
Line 18	G=^{3}√(0.5(-E+√(F)))	[ALPHA] [ln G] [ALPHA] [ENTER] [2nd] [∛] [• ;] [5 U] [( ) N] [(−) M] [ALPHA] [DATA E] [+] [√ K] [ALPHA] [log F] ▶ ▶ ▶ [ENTER]
Line 19	H=^{3}√(0.5(-E–√(F)))	[ALPHA] [sin H] [ALPHA] [ENTER] [2nd] [∛] [• ;] [5 U] [( ) N] [(−) M] [ALPHA] [DATA E] [− SPC] [√ K] [ALPHA] [log F] ▶ ▶ ▶ [ENTER]
Line 20	IF(F==0)THEN{	[2nd] [PROG] [ENTER] [ALPHA] [log F] [2nd] [==] [0 ,] ▶ ▶ [ENTER]
Line 21	PRINT “3 REAL”	[2nd] [PROG] ▲ ▲ ▲ [ENTER] [ALPHA] [EXP] [3 Z] [2nd] [ALPHA] [− SPC] [9 R] [DATA E] [SAVE A] [x^2 L] [ALPHA] ▶ [ENTER]
Line 22	SLEEP(1)	[2nd] [PROG] ▲ [0 ,] [1 X] ▶ [ENTER]
Line 23	X=-((G+H)/2+A/3)	[ALPHA] [1 X] [ALPHA] [ENTER] [(−) M] [( ) N] [( ) N] [ALPHA] [ln G] [+] [ALPHA] [sin H] ▶ [÷ S] [2 Y] [+] [ALPHA] [SAVE A] [÷ S] [3 Z] ▶ [ENTER]
Line 24	Y=X	[ALPHA] [2 Y] [ALPHA] [ENTER] [ALPHA] [1 X] [ENTER]
Line 25	Z=G+H–A/3	[ALPHA] [3 Z] [ALPHA] [ENTER] [ALPHA] [ln G] [+] [ALPHA] [sin H] [− SPC] [ALPHA] [SAVE A] [÷ S] [3 Z] [ENTER]
Line 26	PRINT X; ◢	[2nd] [PROG] ▲ ▲ ▲ [ENTER] [ALPHA] [1 X] ▶ [ALPHA] [+] [ENTER]
Line 27	PRINT Y; ◢	[2nd] [PROG] ▲ ▲ ▲ [ENTER] [ALPHA] [2 Y] ▶ [ALPHA] [+] [ENTER]
Line 28	PRINT Z	[2nd] [PROG] ▲ ▲ ▲ [ENTER] [ALPHA] [3 Z] [ENTER]
Line 29	END }	[2nd] [PROG] ▲ [1 X] [ALPHA] ▶❙ [ENTER]
Line 30	PRINT “1 REAL”	[2nd] [PROG] ▲ ▲ ▲ [ENTER] [ALPHA] [EXP] [1 X] [2nd] [ALPHA] [− SPC] [9 R] [DATA E] [SAVE A] [x^2 L] [ALPHA] ▶ [ENTER]
Line 31	SLEEP(1)	[2nd] [PROG] ▲ [0 ,] [1 X] ▶
Line 32	X=G+H–A/3	[ALPHA] [1 X] [ALPHA] [ENTER] [ALPHA] [ln G] [+] [ALPHA] [sin H] [− SPC] [ALPHA] [SAVE A] [÷ S] [3 Z] [ENTER]
Line 33	Y=-((G+H)/2+A/3)	[ALPHA] [2 Y] [ALPHA] [ENTER] [(−) M] [( ) N] [( ) N] [ALPHA] [ln G] [+] [ALPHA] [sin H] ▶ [÷ S] [2 Y] [+] [ALPHA] [SAVE A] [÷ S] [3 Z] ▶ [ENTER]

Line 34	Z=√(3)(G–H)/2	[ALPHA] [3 Z] [ALPHA] [ENTER] [√ K] [3 Z] ▶ [( ) N] [ALPHA] [ln G] [– SPC] [ALPHA] [sin H] ▶ [÷ S] [2 Y] [ENTER]
Line 35	PRINT X; ◢	[2nd] [PROG] ▲ ▲ ▲ [ENTER] [ALPHA] [1 X] ▶ [ALPHA] [+ ◢] [ENTER]
Line 36	PRINT Y,"+/ –",Z,"I"; ◢	[2nd] [PROG] ▲ ▲ ▲ [ENTER] [ALPHA] [2 Y] [ALPHA] [0 ,] [ALPHA] [EXP "] [+ ◢] [÷ S] [– SPC] ▶ [ALPHA] [0 ,] [ALPHA] [3 Z] [ALPHA] [0 ,] [ALPHA] [EXP "] [ALPHA] [cos I] ▶ ▶ [ALPHA] [+ ◢]

When the program runs, it prompts for the values for a, b and c of the cubic equation $x^3 + ax^2 + bx + c = 0$. Then the program briefly displays the message "1 REAL" or " 3 REAL" according to the number of real roots. If there are 3 real roots, they are displayed one by one so that you can jot them down – press [ENTER] to display the next root. If there's one real root then it is displayed and after pressing [ENTER] the complex conjugate pair is displayed thus: a +/–bi in the entry line. Use the ◀ , ▶ , [ALPHA] ◀◀ and [ALPHA] ▶▶ keys to scroll the entry line.

Example 3: Find the roots of $x^3 + 6x^2 + 3x - 10 = 0$

Solution: First of all, make sure the angle mode is DEG, since the cosine functions included in the program must work in degrees. Once the above program has been entered, enter MAIN mode ([MODE] [0 ,]) and run the program by pressing [PROG] , the corresponding program number and [ENTER] . "A= ◀ " appears in the display. Press [6 V] and [ENTER] . It now asks for the value of B. Press [3 Z] and [ENTER] . Finally, enter C: press [(–) M] [1 X] [0 ,] and [ENTER] .when the message "C= ◀ " is displayed. For one second the message "3 REAL" appears and then the first root is displayed: 1. When ready, press [ENTER] to display the second one, and once again [ENTER] to display the third one.

Answer: The roots are 1, -5 and -2.

(Note: Refer to the HP 9g learning module *Graphing Functions – Part 2* for a way of finding estimates of real roots graphically).

Example 4: Find the roots of $x^3 - 11x^2 + 44x - 34 = 0$

Solution: If –2 is still displayed in the entry line, from the previous example, just press [ENTER] to restart the program from the beginning. A, B and C are then requested as before. Press:

[(–) M] [1 X] [1 X] [ENTER] [4 T] [4 T] [ENTER] [(–) M] [3 Z] [4 T] [ENTER]

The displayed message is now "1 REAL". Press [ENTER] to display the real root, and then [ENTER] again to display the complex conjugate pair, which is displayed as $5 + /- 3I$.

Answer: The roots are 1 and the Gaussian integers $5 \pm 3i$.